КОНДЕНСАЦИЯЛАНҒАН КҮЙДІҢ ФИЗИКАСЫ
ФИЗИКА КОНДЕНСИРОВАННОГО СОСТОЯНИЯ
PHYSICS OF THE CONDENSED MATTER

DOI: 10.31489/2019Ph4/8-14

UDC 53.012+52.336

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Solution of the deformed Schwarzschild metric by the Yang-Baxter equation

In this article, an open-closed string map formulated by Seiberg & Whitten was used to solve problems of generalized supergravity, including the deformed Schwarzschild metric. For this task, found analytical supergravity solution (deformed metric and NSNS (Neveu – Schwarz) two-form $B_{\mu\nu}$-field). The solution was obtained from antisymmetric bivector constructed from antisymmetric products of Killing vectors used as components of the equation of motion. In the problem under consideration, the equations of motion are the CYBE (classical Yang-Baxter equation), whose general solution can be obtained using the r-matrix. As a result, for the deformed metric, the Hamilton — Jacobi equation is obtained, the particle motion on the plane is studied, with $\theta = \pi/2$. So, we obtained several analytical solutions for the function $r(\varphi)$, $\varphi(r)$. Since these results are very voluminous for representations, we present the schedule the test particle from the function $r(\varphi)$, which shows the centrally- symmetric motion of the particle in the Schwarchild field. As a continuation of this work, it is possible to obtain a numerical solution for a function $r(t)$, that has a complex integral for the analytical solution of this problem. The theoretical meaning of the work is that CYBE derives from the equation of motion of the theory of gravity, thereby reducing the problem of determining the r-matrix, which is a CYBE solution for generalized supergravity.

Keywords: Classical Yang-Baxter equation, Hamilton-Jacobi equation, open-closed string map, supergravity, Killing vectors, B-field, NS sector, TsT transform, antisymmetric bivector.

Introduction

Yang-Baxter string sigma — models provide a systematic approach of deforming the geometry of coset classes, like $AdS_p \times S^p$ while maintaining the integrability of the σ-model. It was demonstrated that the Yang-Baxter deformation in the target space for any geometry can be considered as an open-closed map of the string. Considering the geometry as a bivector and an isometry group, which is determined by the linear combination of antisymmetric products of Killing vectors, is possible to get the equations of motion generalized supergravity, which is equivalent to the classical Yang-Baxter equation (CYBE) [1].

Integrable deformations of σ-models [2, 3] is possible to apply to string σ-model and AdS/CFT geometries [4, 5]. So were investigated noncommutative [6–8] and marginal deformations [9, 10] of AdS/CFT geometries: as a part of a larger family of integrable deformations of $AdS_p \times S^p$ geometries in which the deformation is specified using the r-matrix for solving the CYBE.

In this work, we will try using the open-closed string map [11] as a simple, effective method for solution.

Let's take «open string data», which produced «closed string data» as a new metric g and B-field after inverting single matrix. Dilaton is determined by T-duality invariant and Killing vector I is a the divergence of antisymmetric bivector $\Theta^{\mu\nu}$ [12]:

$$\nabla_\mu \Theta^{\mu\nu} = I^\nu.$$

Building the open-closed string map of Seiberg & Witten [13], which were write as:

$$(g+B)_{\mu\nu}=(G^{\mu\nu}+\Theta^{\mu\nu})^{-1}.$$

Here $(g,B),(G,\Theta)$ are respectively closed string and open string fields. The metrics g,G are symmetric and B,Θ are antisymmetric counterparts.

Connection of the deformed solution ($g_{\mu\nu},B_{\mu\nu},\varphi$) with the original solution ($G_{\mu\nu},\Theta^{\mu\nu},\Phi$) of NS sector has the form [14]:

$$\begin{gathered}g_{\mu\nu}=(G^{-1}-\Theta\cdot G\cdot\Theta)^{-1}_{\mu\nu};\\ B_{\mu\nu}=-(G^{-1}-\Theta)^{-1}\cdot\Theta\cdot(G^{-1}+\Theta)^{-1};\\ \varphi=\Phi-\frac{1}{2}\ln(\det(1+G\cdot\Theta)).\end{gathered}\tag{1}$$

In the above $\cdot$ denotes matrix multiplication and G and Θ are to be viewed as two matrices. The indices on Θ are lowered and raised by the metric G.

r-matrix solutions to the CYBE take the form [14]

$$r=\frac{1}{2}r^{ij}T_i\wedge T_j,$$

where T_i are elements of the Lie algebra, $T_i\in g$. Here the bi-Killing structure of Θ is the r-matrix written in the basis of Killing vectors.

For the object of the study we take the Schwarzschild black hole metric

$$ds^2=-\left(1-\frac{2m}{r}\right)dt^2+\frac{dr^2}{1-\frac{2m}{r}}+r^2\left(d\varphi^2+\sin^2\varphi d\theta^2\right),$$

it is convenient to write the metric in matrix in the form

$$G=\begin{vmatrix}-a&0&0&0\\0&1/a&0&0\\0&0&r^2&0\\0&0&0&K^2r^2\end{vmatrix}.$$

We choose Θ to an antisymmetric product of Killing vectors:

$$\begin{aligned}&\Theta^{t\phi}=-e\cos\theta+\lambda\sin\theta;\\&\Theta^{t\theta}=\delta+\cot\phi(e\sin\theta+\lambda\cos\theta);\\&\Theta^{\phi\theta}=\alpha\cos\theta-\beta\cot\phi+\gamma\sin\theta;\end{aligned}$$

rewrite in matrix form

$$\Theta=\begin{vmatrix}0&0&-D&\delta+Z\cdot M\\0&0&0&0\\D&0&0&N\\-\delta-Z\cdot M&0&-N&0\end{vmatrix}.$$

Made such replacements to simplify the calculation

$$\begin{aligned}&a=1-\frac{2m}{r};\\&K=\sin\varphi;\\&D=e\cdot\cos\theta-\lambda\cdot\sin\theta;\\&Z=\cot\varphi;\\&M=e\cdot\sin\theta+\lambda\cdot\cos\theta;\\&N=\alpha\cdot\cos\theta-\beta\cdot\cot\varphi+\gamma\cdot\sin\theta.\end{aligned}\tag{2}$$

Using formula (1) we find the supergravity solutions (deformed metric, B-field)

$$ds^2=\frac{1}{a\Omega}\Big[(K^2N^2r^4a^2+a^2)dt^2+dr^2+\big(K^2M^2Z^2a^2r^4+2K^2MZa^2\delta r^4+$$
$$+K^2a^2\delta^2r^4-r^2a\big)d\varphi^2+(D^2K^2a^2r^4-r^2K^2a)d\chi^2-(K^2MNZa^2r^4+K^2Na^2\delta r^4)(dtd\varphi+d\varphi dt)- \quad (3)$$
$$-Dr^4Na^2K^2(dtd\theta+d\theta dt)+(DK^2MZa^2r^4+DK^2a^2\delta r^4)(d\varphi d\theta+d\theta d\varphi)\Big],$$

the components of the metric tensor is more convenient to write as a matrix

$$g=\begin{vmatrix} \frac{a(K^2N^2r^4+1)}{\Omega} & 0 & -\frac{a(MZ+\delta)K^2r^4N}{\Omega} & -\frac{Dr^4NaK^2}{\Omega} \\ 0 & 1/a & 0 & 0 \\ -\frac{a(MZ+\delta)K^2r^4N}{\Omega} & 0 & \frac{(K^2M^2Z^2ar^2+2K^2MZa\delta r^2+K^2a\delta^2r^2-1)r^2}{\Omega} & \frac{r^4Da(MZ+\delta)K^2}{\Omega} \\ -\frac{Dr^4NaK^2}{\Omega} & 0 & \frac{r^4Da(MZ+\delta)K^2}{\Omega} & \frac{(D^2ar^2-1)r^2K^2}{\Omega} \end{vmatrix}, \quad (4)$$

where

$$\Omega=K^2M^2Z^2ar^2+2K^2MZa\delta r^2-K^2N^2r^4+K^2a\delta^2r^2+D^2ar^2-1. \quad (5)$$

$$B_{\mu\nu}=\frac{r^2aD}{\Omega}dt\wedge d\varphi+\frac{r^4K^2N}{\Omega}d\varphi\wedge d\theta+\frac{r^2aK^2(MZ+\delta)}{\Omega}d\theta\wedge dt; \quad (6)$$

$$B=\begin{vmatrix} 0 & 0 & \frac{r^2aD}{\Omega} & -\frac{r^2aK^2(MZ+\delta)}{\Omega} \\ 0 & 0 & 0 & 0 \\ -\frac{r^2aD}{\Omega} & 0 & 0 & \frac{r^4K^2N}{\Omega} \\ \frac{r^2aK^2(MZ+\delta)}{\Omega} & 0 & -\frac{r^4K^2N}{\Omega} & 0 \end{vmatrix}.$$

To determine the trajectory of a particle, use the Hamilton-Jacobi equation [15]

$$g^{ik}\frac{ds}{dx^i}\cdot\frac{ds}{dx^k}-m^2c^2=0. \quad (7)$$

With the metric tensor (4), eq. (7) takes the following form

$$g^{tt}\left(\frac{ds}{dt}\right)^2+g^{rr}\left(\frac{ds}{dr}\right)^2+g^{\varphi\varphi}\left(\frac{ds}{d\phi}\right)^2+g^{\theta\theta}\left(\frac{ds}{d\theta}\right)^2+$$
$$+g^{t\varphi}\frac{ds}{dt}\frac{ds}{d\varphi}+g^{\phi t}\frac{ds}{d\varphi}\frac{ds}{dt}+g^{t\theta}\frac{ds}{dt}\frac{ds}{d\theta}+g^{\theta t}\frac{ds}{d\theta}\frac{ds}{dt}+g^{\varphi\theta}\frac{ds}{d\varphi}\frac{ds}{d\theta}+g^{\theta\varphi}\frac{ds}{d\theta}\frac{ds}{d\varphi}-m^2c^2=0, \quad (8)$$

where $g^{t\phi}=g^{\phi t},g^{t\theta}=g^{\theta t},g^{\phi\theta}=g^{\theta\phi}$.

By the general rules, the solution of the Hamilton-Jacobi equation s is sought in the form [16]

$$s=-\varepsilon t+M\varphi+s_r(r), \quad (9)$$

where ε — energy and M — angular momentum are constant.

Substituting eq. (9) into eq. (8) we found

$$\frac{d(s_r(r))}{dr}=\frac{\sqrt{m^2c^2-(\frac{\varepsilon}{c})^2g^{tt}+\frac{2M\varepsilon}{c}g^{t\varphi}-M^2g^{\varphi\varphi}}}{g^{rr}}. \quad (10)$$

Lowering the indices of the components of the metric tensor, the eq. (10) is written

$$s_r(r)=\int\sqrt{\left(m^2c^2-\frac{\varepsilon^2}{c^2g_{tt}}+\frac{2M\varepsilon}{cg_{\varphi t}}-\frac{M^2}{g_{\varphi\varphi}}\right)g_{rr}}dr, \quad (11)$$

there we use the formula lowering and raising the indices of the metric tensor $g_{jk} = \frac{\delta_k^i}{g^{ij}}$:

$$g_{tt} = \frac{1}{g^{tt}},\ g_{t\varphi} = \frac{1}{g^{\varphi t}},\ g_{\varphi\varphi} = \frac{1}{g^{\varphi\varphi}},\ g_{rr} = \frac{1}{g^{rr}}. \tag{12}$$

Using expression $\frac{ds}{dM} = const$, from eq. (9) we can obtain $\Delta\varphi$

$$\Delta\varphi = const - \frac{d(s_r(r))}{dM}, \tag{13}$$

substituting eq. (11) to (13) we will result in the following eq.

$$\Delta\varphi = const - \int \frac{\left(\frac{\varepsilon}{cg_{\varphi t}} - \frac{M}{g_{\varphi\varphi}}\right)\sqrt{g_{rr}}}{\sqrt{m^2c^2 - \frac{\varepsilon^2}{c^2 g_{tt}} + \frac{2M\varepsilon}{cg_{\varphi t}} - \frac{M^2}{g_{\varphi\varphi}}}} dr. \tag{14}$$

Since the integral in equation (14) is difficult to solve we take the derivative with respect to r from both sides of the eq.

$$\frac{d\varphi}{dr} = -\frac{\left(\frac{\varepsilon}{cg_{\varphi t}} - \frac{M}{g_{\varphi\varphi}}\right)\sqrt{g_{rr}}}{\sqrt{m^2c^2 - \frac{\varepsilon^2}{c^2 g_{tt}} + \frac{2M\varepsilon}{cg_{\varphi t}} - \frac{M^2}{g_{\varphi\varphi}}}} = 0; \tag{15}$$

that numerator of equation (15) can be reduced to zero

$$-\left(\frac{\varepsilon}{cg_{\varphi t}} - \frac{M}{g_{\varphi\varphi}}\right)\sqrt{g_{rr}} = 0. \tag{16}$$

To simplify the problem, we consider the case when the motion occurs in one plane, that is, we can take it $\theta = \frac{\pi}{2}$, since the field is central, then the replacements in eq. (2) will take this form

$$\begin{aligned} a &= 1 - \frac{2m}{r}; \\ K &= \sin\varphi; \\ D &= -\lambda; \\ Z &= \cot\varphi; \\ M &= e; \\ N &= -\beta\cdot\cot\varphi + \gamma. \end{aligned} \tag{17}$$

Substituting these expressions (17) into components $g_{\varphi\varphi}, g_{\varphi t}, g_{rr}$, getting two equations (18), (19)

$$\frac{\varepsilon}{c(1-2m/r)(r^4\sin^2\varphi(\gamma-\beta\cot\varphi)^2+1)} + \\ + \frac{M}{(e^2r^2\sin^2\varphi\cot^2\varphi(1-2m/r) + 2e\delta r^2\sin^2\varphi\cot\varphi(1-2m/r) + \delta^2r^2\sin^2\varphi(1-2m/r)-1)r^2} = 0, \tag{18}$$

from this eq. (18) we get: $r(\varphi)$ have 2 solutions, $\varphi(r)$ have 5 solutions.

$$g_{rr}\cdot\Omega = r^2e^2\sin^2\varphi\cot^2\varphi + 2e\delta\sin^2\varphi\cot\varphi r^2 + \delta^2\sin^2\varphi r^2 + \lambda^2r^2 - \frac{1+\sin^2\varphi(\gamma-\beta\cot\varphi)^2r^4}{(1-2m/r)} = 0, \tag{19}$$

from this eq. (19) we get: $r(\varphi)$ have 5 solutions, $\varphi(r)$ have 8 solutions.

So we got some non-zero solutions that are complex and lengthy to write. We can demonstrate one of the result graphically in Figure.

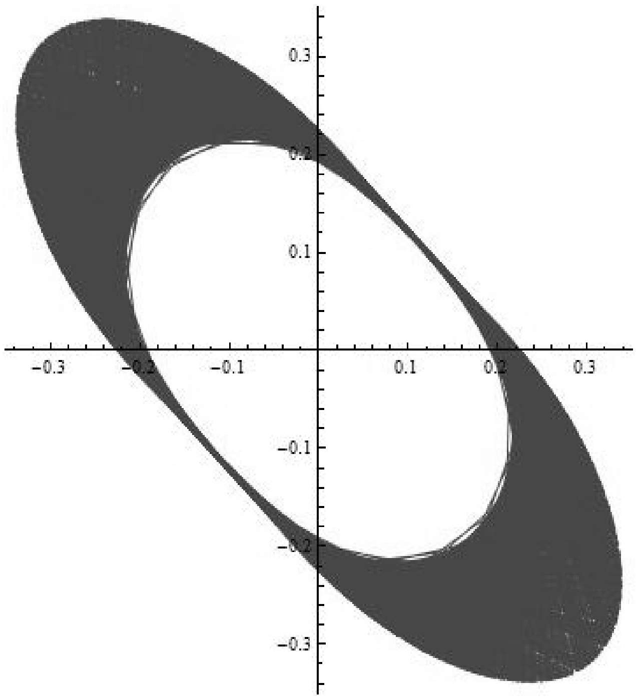

Figure. Movement of test particle in deformed Schwarzschild field, when $\alpha, \beta, \gamma, \delta, \lambda, e, m, \varepsilon, M, c = 1$. The graph is built from a function $r(\varphi)$

Conclusion

Let us review what has been achieved. We consider $AdS_2 \times S^2$ and Schwarzschild spacetimes and found supergravity solutions for it, using an open-closed string map formulated by Seiberg & Whitten.The solution is determined by a bivector, constructed from antisymmetric products of Killing vectors used as components of the equation of motion. The main point of the work is that CYBE emerges from the equations of motions of a gravity theory, thereby simplifying the problem of determining the r-matrix solution to the CYBE.

As a result, for the deformed metric, the Hamilton – Jacobi equation is obtained, the particle motion on the plane is studied, with $\theta = \frac{\pi}{2}$. We obtained some analytic exact solutions for the functions $r(\varphi)$ and $\varphi(r)$, which are very long to write, therefore, we will provide only a graph of one of the solution. From the graph of a test particle of function $r(\varphi)$ is concluded: particle moving in a centrally symmetric Schwarzschild field are obtained. It is also easy to yet a solution $r(t)$, which contains an integral, difficult for an analytical solution.

Originally integrability was finded in string models on $AdS_p \times S^q$ (which described are β-deformation). There are also integrable string theories that known as η, λ-deformations, which are considered models on $AdS_p \times S^p$. In the case of η, λ-deformations, the CFT (conformal field theories) construction gives the NS-NS fields. For deformation our model wording for NS-sector, respectively. Following CFT we must construction classical R-matrix for the selected model. The R-matrix is classical solution of CYBE.

Our work is based on the consideration of the integrability of model deformation (in our case, Schwarzschild black hole) geometry $AdS_2 \times S^2$, where the deformation is expressed through the solutions of the R-matrix CYBE. When the model is deformed, symmetries are preserved, which is typical for CYBE.

The Hamilton–Jacobi solutions obtained from deformed metric (3) generated through the open-closed string map give us the particle trajectory in the Schwarzschild black hole in geometry $AdS_2 \times S^2$.

As the form of the particle trajectory is symmetrically different from the results of calculations of the standard Schwarzschild metric. It can be seen from figure 1 that the particle trajectory obtained from the deformed metric will look like a folding ellipsoid spiral, while the standard Schwarzschild metric gives the circle spiral.

For our case, it is deformed (squeezed) more than twice. The diameter perpendicular to it is compressed, approaching the shape of the circle with time.

The work was carried out with the financial support of the Ministry of Education and Science of the Republic of Kazakhstan, Grant No. 0118RK00935.

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А. Мейрамбай, К.К. Ержанов

Янг-Бакстер теңдеуінің әдістері арқылы алынған Шварцшильд деформацияланған метрикасының шешімдері

Мақалада жалпыланған супергравитация есептері үшін, соның ішінде Шварцшильд деформацияланған метрикасы үшін Сейнберг және Уиттонмен жасалған ашық-жабық ішектік карта пайдаланылды. Бұл есеп үшін аналитикалық супергравитациялық шешім алынды (деформацияланған метрика және NSNS (Neveu–Schwarz) екіформалы $B_{\mu\nu}$-өріс). Қозғалыс теңдеуінің компоненттері ретінде қолданылған, Киллинг векторларының антисимметриялық көбейтіндісінен құрастырылған антисимметриялық бивектор пайдаланылды. Қарастырылып отырған есепте қозғалыс теңдеуінің жалпы шешімі *r*-матрицы арқылы алынатын классикалық Янг-Бакстер теңдеуі (КЯБТ) болып табылады. Қорытындылай келгенде, деформацияланған метрика үшін Гамильтон-Якоби теңдеуі алынған, $\theta = \pi/2$ жазықтығында бөлшек қозғалысы зерттелген. Сонымен қатар $r(\varphi)$, $\varphi(r)$ функциялардың бірнеше аналитикалық шешімдері берілген. Бұл мәліметтер өте көлемді болғандықтан, Шварцшильд өрісінде бөлшектің орталық-симметриялы қозғалысын көрсететін $r(\varphi)$ функциясынан алынған сынақ бөлшектің графигі ғана көрсетілді. Осы жұмыстың жалғасы ретінде есепті аналитикалық шешудің күрделі интегралына ие $r(t)$ функциясы үшін сандық шешімді алуға болады. Жұмыстың теориялық мәні классикалық Янг-Бакстер теңдеуі гравитация теориясының қозғалыс теңдеуінен шығып, КЯБТ шешімі үшін жалпыланған супергравитация болып табылатын, *r*-матрицаны анықтау міндетін азайтатынында.

Кілт сөздер: классикалық Янг-Бакстер теңдеуі, Гамильтон-Якоби теңдеуі, ашық-тұйық ішектік карта, супергравитация, Киллинг векторлары, B-өрісі, NS-сектор, TsT түрлендіру, антисимметриялық бивектор.

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Решение деформированной метрики Шварцшильда методами уравнения Янга-Бакстера

В статье для решения задач обобщенной супергравитации, в том числе деформированной метрики Шварцшильда была использована открытая-закрытая струнная карта, сформулированная Сейнбергом и Уиттоном. Для этой задачи было получено аналитическое супергравитационное решение (деформированная метрика и NSNS (Neveu–Schwarz) двухформное $B_{\mu\nu}$-поле) с использованием антисимметричного бивектора, построенного из антисимметричных произведений векторов Киллинга, использованных как компоненты уравнения движения. В рассматриваемой задаче уравнение движения является классическим уравнением Янга-Бакстера (КУЯБ), чье общее решение можно получить с помощью r-матрицы. В итоге, для деформированной метрики получено уравнение Гамильтона–Якоби, исследовано движение частицы на плоскости при $\theta = \pi/2$. Кроме того, даны несколько аналитических решений для функций $r(\varphi)$, $\varphi(r)$. Так как эти данные очень объемны для представлений, продемонстрирован только график пробной частицы от функции $r(\varphi)$, который показывает центрально-симметричное движение частицы в шварцшильдовском поле. Как продолжение этой работы возможно получить численное решение для функции $r(t)$, которое имеет сложный интеграл для аналитического решения задачи. Теоретический смысл работы заключается в том, что классическое уравнение Янга-Бакстера вытекает из уравнения движения теории гравитации, тем самым уменьшая задачу определения r-матрицы, которая является решением КУЯБ для обобщенной супергравитации.

Ключевые слова: классическое уравнение Янга-Бакстера, уравнение Гамильтона-Якоби, открытая-закрытая струнная карта, супергравитация, векторы Киллинга, B-поле, NS-сектор, TsT-преобразования, антисимметричный бивектор.